THERMOMECHANICAL BEHAVIOR OF A MULTICOMPONENT CONTINUOUS MEDIUM INTERACTING WITH AN ELECTROMAGNETIC FIELD

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The behavior of a homogeneous mixture of isotropic continua interacting with an electromagnetic field is discussed. It is assumed that the continuum has a simple memory for strains, temperature, polarization, and magnetization. Constitutive functionals are constructed for the continuum and a theorem on free energy is proved.

We describe the behavior of a mixture of n isotropic conducting continua using a model of a continuous medium with internal degrees of freedom [1, 2], taking account of the effect of the average motion of the microstructure of the medium on its macroscopic behavior. In accord with well-known [3-8] concepts we define a translational velocity vector $\vec{v}$ and a particle rotation tensor $\vec{\vec{\Omega}}$ at each point of the medium. The medium is assumed capable of being polarized and magnetized and has charges, currents, and internal sources of material such as chemical reactions.

We assume that a body force of density $\vec{f}_k$ is applied at each point of a volume V of the continuous material, and force stresses and electromagnetic stresses described by the vectors $\vec{s}_n$ and $\vec{\tau}_n$ respectively act on its surface Σ.

Basic Laws of Mechanics and Thermodynamics. Omitting the mass and momentum balance equations because of their triviality we have the following system of balance equations for motions which are continuous except at a finite number of singularities:

$$\rho\frac{d\vec{\vec{N}}}{dt}=\rho\vec{\vec{D}}+\nabla\cdot\vec{\vec{\vec{Q}}}-\vec{\vec{\Lambda}}+\vec{\vec{\Pi}};$$

$$\rho\frac{du}{dt}=-\nabla\cdot\vec{J}_q+\vec{\vec{\Pi}}\cdot\cdot\nabla\vec{v}+\vec{i}\cdot\vec{E}+\rho\vec{E}\cdot\frac{d\vec{p}}{dt}$$

$$+\rho\vec{H}\cdot\frac{d\vec{\mu}}{dt}+\rho b+\sum_k\vec{j}_k\cdot\vec{f}_k+(\vec{\vec{\Lambda}}-\vec{\vec{\Pi}})\cdot\cdot\vec{\vec{\Omega}}+\vec{\vec{\vec{Q}}}\cdot\cdot\cdot\nabla\vec{\vec{\Omega}}; \tag{1}$$

$$\rho\frac{d\eta}{dt}-\rho\sigma-\nabla\cdot\vec{J}_\eta\geqslant 0.$$

Here $\vec{\vec{N}}$ is the spin inertia tensor [10]; $\vec{\vec{\vec{Q}}}$ is a third rank tensor of the first stress moments, assumed antisymmetric; $\vec{J}_\eta$ is the entropy flux vector. The operations (..) and (...) denote double and triple contraction respectively. We note that Eqs. (1) follow from the basic system of balance equations for continuous media [9, 10], and include the balance equation for first stress moments known from [10] where a detailed explanation of the various terms of the equation is given. The second of Eqs. (1) represents the first law of thermodynamics taking account of polarization and magnetization, and the third expresses the Clausius—Duhem inequality [10]. Assuming that in the absence of microstretching of particles $\vec{\vec{\vec{Q}}}$ and $\vec{\vec{\Omega}}$ are subjected to antisymmetric boundary conditions we multiply the first of Eqs. (1) by the Levi—Civita alternating tensor and obtain the balance equation for first moments valid for a narrower class of continua than the original, but more widely known in practice:

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$$\rho J \frac{d\vec{\omega}}{dt} = \nabla \cdot \vec{\vec{M}} + \vec{h} + \vec{\Pi}^{(a)}. \tag{2}$$

Equation (2) is the balance equation for internal angular momenta, a possible consequence of the rotational motion of the particles comprising the system. The right-hand side of (2) contains the moments of the distributed body and surface forces and the antisymmetric part of the force stress tensor expressed in terms of the axial vector $\vec{\Pi}^{(a)}$.

The latter two of Eqs. (1) together with the conditions [11]

$$\vec{J}_\eta = \vec{J}_\eta^{(1)} + \vec{J}_\eta^{(2)} = -\frac{\vec{J}_q - \Sigma\varphi_k \vec{j}_k}{\Theta}; \tag{3}$$

$$\rho\sigma = \rho\sigma_1 + \frac{\rho b}{\Theta}, \tag{4}$$

lead to the following local Clausius—Duhem inequality

$$\begin{aligned} &\rho\left(\frac{d\eta}{dt} - \frac{1}{\Theta}\frac{du}{dt}\right) + \frac{1}{\Theta}\vec{\vec{\Pi}} \cdot\cdot \nabla\vec{v} + \frac{1}{\Theta}\vec{i}\cdot\vec{E} - \rho\sigma_1 \\ &+ \frac{\rho}{\Theta}(\vec{E}_0 - \vec{E}) \cdot \frac{d\vec{p}}{dt} + \frac{\rho}{\Theta}(\vec{H}_0 - \vec{H}) \cdot \frac{d\vec{\mu}}{dt} - \nabla \cdot \vec{J}_\eta^{(2)} - \frac{1}{\Theta^2}\vec{J}_q \cdot \nabla\Theta \\ &+ \frac{1}{\Theta}\vec{\vec{Q}} \cdot\cdot \nabla\vec{\vec{\Omega}} + \frac{1}{\Theta}(\vec{\vec{\Lambda}} - \vec{\vec{\Pi}}) \cdot\cdot \vec{\vec{\Omega}} + \frac{1}{\Theta}\sum_k \vec{j}_k \cdot \vec{f}_k \geqslant 0. \end{aligned} \tag{5}$$

If condition (2) is taken into account, we can write (5) in the form

$$\begin{aligned} &\rho\left(\frac{d\eta}{dt} - \frac{1}{\Theta}\frac{du}{dt}\right) + \frac{1}{\Theta}\vec{\vec{\Pi}} \cdot\cdot \nabla\vec{v} + \frac{1}{\Theta}\vec{i}\cdot\vec{E} - \rho\sigma_1 \\ &+ \frac{\rho}{\Theta}(\vec{E}_0 - \vec{E}) \cdot \frac{d\vec{p}}{dt} + \frac{\rho}{\Theta}(\vec{H}_0 - \vec{H}) \cdot \frac{d\vec{\mu}}{dt} + \frac{1}{\Theta}\vec{\vec{M}} \cdot\cdot \nabla\vec{\omega} \\ &+ \frac{1}{\Theta}\vec{\vec{\Pi}} \times \cdot \vec{\vec{I}} \cdot \vec{\omega} + \frac{1}{\Theta}\sum_k \vec{j}_k \cdot \vec{f}_k - \frac{1}{\Theta^2}\vec{J}_q \cdot \nabla\Theta - \nabla \cdot \vec{J}_\eta^{(2)} \geqslant 0. \end{aligned} \tag{6}$$

Here σ_1 is the strength of the internal entropy source arising from factors other than the energy sources b; $\vec{E}_0$ and $\vec{H}_0$ are the intensities of the electric and magnetic fields for an equilibrium transformation; Θ is the temperature; $\times\cdot$ means that the first factors of the dyads are multiplied vectorially and the second scalarly.

Basic Laws of Electrodynamics. The equations of conservation of charge, the electromagnetic field, the conservation of momentum and energy density of the electromagnetic field, and the expressions for the ponderomotive force and the electromagnetic stress tensor are taken from [12, 13].

Construction of Modified Axioms. Basic Theorem of Constitutive Theory for Materials with a Simple Memory. The construction of constitutive functionals requires the use of the basic axioms of constitutive theory first proposed by A. Eringen [14, 15] for thermomechanical materials. However, these basic axioms must be modified for the present investigation to include nonsimple thermodynamic processes for a model of a polarizable and magnetizable multicomponent continuum in electromagnetic fields.

The axiom of causality must be modified for our conditions so that the independent variables include not only motion and temperature, but also microrotations, concentrations, and the polarization and magnetization per unit mass $\vec{p}$ and $\vec{\mu}$.

The axiom of determinism must be modified for the nonsimple thermomechanical processes under discussion so that the constitutive functionals for $\vec{\vec{\Pi}}$, $\vec{\vec{M}}$, $\vec{J}_q$, η, Φ, $\vec{E}$, $\vec{H}$, and $\vec{i}$ will depend not only on the history of the motion and temperature, but also on the history of the change of concentrations, and the history of rotations, polarization, and magnetization.

We limit ourselves to the investigation of the practically important case of a simple material with a simple thermomechanical and electromagnetic memory. By a simple material we mean a material whose constitutive functionals depend on gradients of no higher than the first order. A material with a simple memory is one for which the constitutive functionals depend only on the first time derivatives of their arguments. Thus in the following discussion the results will be limited to continua composed of media whose constitutive functionals, by the axiom of objectivity, depend on the following arguments:

$$\rho^{-1};\ \Theta;\ \vec{p};\ \vec{\mu};\ c_a;\ \vec{v};\ \nabla\vec{r};\ (\nabla\vec{v})^s;\ \nabla\times\vec{v}-2\vec{\omega};\ \nabla\vec{\omega};\ \nabla c_a;\ \nabla\Theta;\ \frac{d\Theta}{dt};$$
$$\frac{d(\nabla\Theta)}{dt};\ \rho\vec{p}^*;\ \rho\vec{\mu}^*;\ \nabla\vec{p};\ \nabla\vec{\mu}. \tag{7}$$

Here ∇ is the del operator; $(\nabla\vec{v})^S$ is the symmetric part of the displacement velocity gradient tensor

$$\vec{p}=\rho^{-1}\vec{P};\ \vec{\mu}=\rho^{-1}\vec{M};\ \rho\vec{p}^*=\frac{d\vec{P}}{dt}+\nabla\cdot(\vec{v}\vec{P})-\vec{\omega}\times\vec{P};$$

$$\rho\vec{\mu}^*=\frac{d\vec{M}}{dt}+\nabla\cdot(\vec{v}\vec{M})-\vec{\omega}\times\vec{M},$$

$\times$ denotes vector multiplication.

We exclude simple memory of concentration on the basis of the axiom of admissibility, since the time derivatives of c_a can be found from the equation of continuity.

On the basis of the axiom of equipresence we assume that the remaining "causes," i.e., the dependent variables $\vec{\vec{M}}$, $\vec{J}_q$, η, Φ, $\vec{E}$, $\vec{H}$, and $\vec{i}$ will be functions of the same set of arguments. Consequently on the basis of the second of Eqs. (1) we have for the Helmholtz free energy $\Phi = u-\Theta\eta$

$$\begin{aligned}
&-\frac{\rho}{\Theta}\Bigg[\frac{\partial\Phi}{\partial(\nabla v)^s_{ij}}\frac{d(\nabla v)^s_{ij}}{dt}-\frac{1}{\rho^2}\frac{\partial\Phi}{\partial\rho^{-1}}\frac{d\rho}{dt}+\frac{\partial\Phi}{\partial\Theta}\frac{d\Theta}{dt}\\
&+\frac{\partial\Phi}{\partial p_i}\frac{dp_i}{dt}+\frac{\partial\Phi}{\partial\mu_i}\frac{d\mu_i}{dt}+\frac{\partial\Phi}{\partial r_{k,i}}\frac{dr_{k,i}}{dt}+\sum_i\frac{\partial\Phi}{\partial c_i}\frac{dc_i}{dt}\\
&+\frac{\partial\Phi}{\partial(\nabla\times v-2\omega)_i}\frac{d(\nabla\times v-2\omega)_i}{dt}+\frac{\partial\Phi}{\partial(\nabla\omega)_{ij}}\frac{d(\nabla\omega)_{ij}}{dt}+\frac{\partial\Phi}{\partial v_i}\frac{dv_i}{dt}\\
&+\sum_i\frac{\partial\Phi}{\partial(\nabla c_i)_j}\frac{d(\nabla c_i)_j}{dt}+\frac{\partial\Phi}{\partial(\nabla\Theta)_i}\frac{d(\nabla\Theta)_i}{dt}+\frac{\partial\Phi}{\partial\dot\Theta}\frac{d^2\Theta}{dt^2}\\
&+\frac{\partial\Phi}{\partial(\nabla\dot\Theta)_i}\frac{d^2}{dt^2}\nabla\Theta_i+\frac{\partial\Phi}{\partial(\rho p^*)_i}\frac{d(\rho p^*)_i}{dt}+\frac{\partial\Phi}{\partial(\rho\mu^*)_i}\frac{d(\rho\mu^*)_i}{dt}\\
&+\frac{\partial\Phi}{\partial(\nabla p)_{ij}}\frac{d(\nabla p)_{ij}}{dt}+\frac{\partial\Phi}{\partial(\nabla\mu)_{ij}}\frac{d(\nabla\mu)_{ij}}{dt}+\eta\frac{d\Theta}{dt}\Bigg]+\frac{1}{\Theta}\vec{\vec{\Pi}}\cdot\cdot\nabla\vec{v}\\
&+\frac{1}{\Theta}\vec{i}\cdot\vec{E}+\frac{\rho}{\Theta}(\vec{E}_0-\vec{E})\cdot\frac{d\vec{p}}{dt}+\frac{\rho}{\Theta}(\vec{H}_0-\vec{H})\cdot\frac{d\vec{\mu}}{dt}-\rho\sigma_1-\nabla\cdot\vec{J}^{(2)}_\eta\\
&+\frac{1}{\Theta}\vec{\vec{M}}\cdot\cdot\nabla\vec{\omega}+\frac{1}{\Theta}\vec{\vec{\Pi}}\times\cdot\vec{I}\cdot\vec{\omega}+\frac{1}{\Theta}\sum_k\vec{j}_k\cdot\vec{f}_k-\frac{1}{\Theta^2}\vec{J}_q\cdot\nabla\Theta\geqslant 0.
\end{aligned}\tag{8}$$

Since (7) shows that the free energy does not depend on the time derivatives of the gradients of $\vec{\omega}$, c_a, $\vec{p}$, and $\vec{\mu}$, and does not depend on the second time derivatives of the temperature and the temperature gradient, and also does not depend on the time derivatives of $\rho\vec{p}^*$ and $\rho\vec{\mu}^*$, and since inequality (8) is linear in these derivatives, the necessary and sufficient conditions for (8) to hold for any independent variations of these derivatives are, according to [16], the relations

$$\eta=-\frac{\partial\Phi}{\partial\Theta};\ \frac{\partial\Phi}{\partial(\nabla v)^s_{ij}}=\frac{\partial\Phi}{\partial(\nabla\times v-2\omega)_i}=\frac{\partial\Phi}{\partial(\nabla\omega)_{ij}}=\frac{\partial\Phi}{\partial\dot\Theta}$$
$$=\sum_i\frac{\partial\Phi}{\partial(\nabla c_i)_j}=\frac{\partial\Phi}{\partial(\nabla\dot\Theta)_i}=\frac{\partial\Phi}{\partial(\rho p^*)_i}=\frac{\partial\Phi}{\partial(\rho\mu^*)_i}=\frac{\partial\Phi}{\partial(\nabla p)_{ij}}=\frac{\partial\Phi}{\partial(\nabla\mu)_{ij}}=\frac{\partial\Phi}{\partial v_i}=\frac{\partial\Phi}{\partial\omega_i}=0. \tag{9}$$

Consequently the free energy Φ is a function of only the following variables: ρ^{-1}, Θ, $\nabla\Theta$, c_a, $\nabla\vec{r}$, $\vec{p}$, and $\vec{\mu}$.

We resolve the stress into elastic and dissipative parts:

$$\vec{\vec{\Pi}}={}_y\vec{\vec{\Pi}}(\rho^{-1};\ \Theta;\ \nabla\Theta;\ c_i;\ \nabla\vec{r};\ \vec{p};\ \vec{\mu})+{}_D\vec{\vec{\Pi}}\Big(\rho^{-1};\ \Theta;\ \vec{p};\ \vec{\mu};\ c_i;\ \vec{v};\ \nabla\vec{r};$$

$$(\nabla\vec{v})^s;\ \nabla\times\vec{v}-2\vec{\omega};\ \nabla\vec{\omega};\ \nabla c_i;\ \nabla\Theta;\ \frac{d\Theta}{dt};$$
$$\frac{d(\nabla\Theta)}{dt};\ \rho\vec{p}^*;\ \rho\vec{\mu}^*;\ \nabla\vec{p};\ \nabla\vec{\mu}\Big). \quad (10)$$

We resolve η into two parts in the same way:

$$\eta = {}_y\eta(\rho^{-1};\ \Theta;\ \nabla\Theta;\ c_i;\ \nabla\vec{r};\ \vec{p};\ \vec{\mu}) + {}_D\eta\Big(\rho^{-1};\ \Theta;\ \vec{p};\ \vec{\mu};\ c_i;\ \vec{v};\ \nabla\vec{r};$$
$$(\nabla\vec{v})^s;\ \nabla\times\vec{v}-2\vec{\omega};\ \nabla\vec{\omega};\ \nabla c_i;\ \nabla\Theta;\ \frac{d\Theta}{dt};\ \frac{d(\nabla\Theta)}{dt};\ \rho\vec{p}^*;\ \rho\vec{\mu}^*;\ \nabla\vec{p};\ \nabla\vec{\mu}\Big). \quad (11)$$

Substituting (9)-(11) into inequality (8) we obtain

$$-\frac{\rho}{\Theta}\left[\frac{\partial\Phi}{\partial r_{i,j}}\frac{dr_{[i,j]}}{dt}+\frac{\partial\Phi}{\partial(\nabla\Theta)_i}\frac{d(\nabla\Theta)_i}{dt}+{}_D\eta\frac{d\Theta}{dt}\right]+\frac{1}{\Theta}{}_D\vec{\vec{\Pi}}\cdot\cdot(\nabla\vec{v})^s$$
$$+\frac{1}{\Theta}\vec{i}\cdot\vec{E}+\frac{1}{\Theta}\vec{\vec{M}}\cdot\cdot\nabla\vec{\omega}+\frac{1}{\Theta}\vec{\vec{\Pi}}\times\cdot\vec{\vec{I}}\cdot\vec{\omega}+\frac{1}{\Theta}\sum_k\vec{j}_k\cdot\vec{f}_k-\frac{1}{\Theta^2}\vec{J}_q\cdot\nabla\Theta-\nabla\cdot\vec{J}_\eta^{(2)}\geq 0, \quad (12)$$

supplemented by the expressions

$${}_y\eta=\eta-{}_D\eta=\frac{\partial\Phi}{\partial\Theta};\ {}_y\Pi_{ij}=\Pi_{ij}-{}_D\Pi_{ij}=\rho\frac{\partial\Phi}{\partial r_{(i,k)}}\frac{dr_{(k,j)}}{dt};$$
$$\frac{\partial\Phi}{\partial c_i}=\frac{\rho\Theta}{l\nu_i};\ \vec{E}_0-\vec{E}=2\frac{\partial\Phi}{\partial J_1}\vec{p};\ \vec{H}_0-\vec{H}=2\frac{\partial\Phi}{\partial J_2}\vec{\mu}. \quad (13)$$

The third of Eqs. (13) is obtained if $\rho\sigma_1$ is related to the chemical reactions in the system by the equation $\rho\sigma_1 = BA/\Theta$, where B is the rate of a chemical reaction and $A = -(\rho\Theta/l\nu_i)(dc_i/dt)$ according to [11]. The factors J_1 and J_2 stand for variations: $J_1 = \vec{p}\cdot\vec{p}$ and $J_2 = \vec{\mu}\cdot\vec{\mu}$.

Thus we have proved the following theorem: the necessary and sufficient condition for the local Clausius—Duhem inequality to be satisfied in simple materials with a simple electromagnetic memory when Eqs. (12) and (13) hold is that the free energy be a function of the arguments:

$$\Phi=\Phi(\rho^{-1};\ \Theta;\ \nabla\Theta;\ c_i;\ \nabla\vec{r};\ \vec{p};\ \vec{\mu})''.$$

The construction of the entropy balance equation and the linear phenomenological equations will be developed in subsequent articles.

NOTATION

$\vec{E}$ is the intensity of the electric field;
$\vec{H}$ is the intensity of the magnetic field;
$\vec{I}$ is the conduction current;
$\vec{p}$ is the polarization per unit mass;
$\vec{\mu}$ is the magnetization per unit mass;
J_q is the heat flux vector;
$\rho = \sum_k\rho_k$ is the density of the mixture;
B_j is the rate of the j-th chemical reaction;
c_i is the concentration of component i;
$\vec{\vec{\Omega}}$ is the rotation tensor;
u is the internal energy density per unit mass;
$\vec{j}_k$ is the diffusion flux of component k;
b is the density of heat sources;
$\vec{\vec{\vec{Q}}}$ is a third rank tensor of the first stress moments;
$\vec{\vec{\Lambda}}$ is a second rank tensor of the average microstresses;
$\vec{\vec{D}}$ is a second rank tensor of the first mass moments;
$\vec{\vec{\Pi}}$ is the force stress tensor;
η is the entropy density per unit mass;

σ is the entropy source strength;

$\vec{\omega}$ is the rotational vector;

J is the scalar moment of inertia;

$\vec{\vec{M}}$ is the couple stress tensor;

$\vec{h}$ is the mass pairs vector;

$\vec{\Pi}^{(a)}$ is an axial vector corresponding to the antisymmetric part of the force stress tensor;

φ_k is the partial specific Gibbs function;

Θ is the temperature.

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